Distance and Midpoint Formulas

ACTIVITY 14

We ♥ Descartes

Lesson 14-1 Distance on the Coordinate Plane

My Notes

Learning Targets:

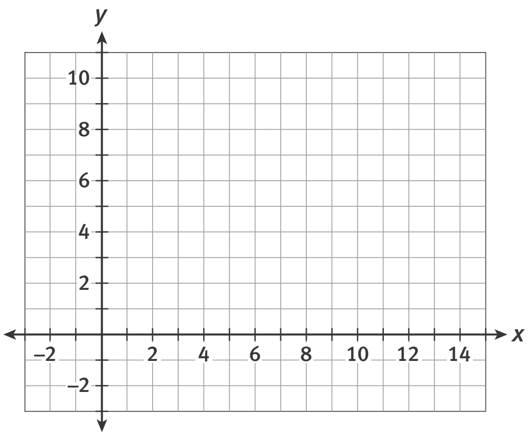
- Derive the Distance Formula.
- Use the Distance Formula to find the distance between two points on the coordinate plane.

SUGGESTED LEARNING STRATEGIES: Create Representations, Simplify the Problem, Think-Pair-Share, Think Aloud, Visualization, Look for a Pattern, Graphic Organizer, Discussion Groups, Identify a Subtask

You have used number lines to determine distance and to locate the midpoint of a segment. A number line is a one-dimensional coordinate system.

In this activity, you will explore the concepts of distance and midpoint on a two-dimensional coordinate system, or coordinate plane.

Use the coordinate plane and follow the steps below to determine the distance between points $P(1, 4)$ and $Q(13, 9)$.



1. **Model with mathematics.** Plot the points $P(1, 4)$ and $Q(13, 9)$ on the coordinate plane. Then draw $\overline{PQ}$.
2. Draw horizontal segment PR and vertical segment QR to create right triangle PQR, with a right angle at vertex R.
3. **Attend to precision.** What are the coordinates of point R?

CONNECT TO HISTORY

The Cartesian coordinate system was developed by the French philosopher and mathematician René Descartes in 1637 as a way to specify the position of a point or object on a plane.

READING MATH

The notation $P(1, 4)$ means point P with coordinates $(1, 4)$ on the coordinate plane.

My Notes

4. a. What is *PR*, the length of the horizontal leg of the right triangle?

b. What is *QR*, the length of the vertical leg of the right triangle?

c. Explain how you determined your answers to parts a and b.

5. Use the Pythagorean theorem to find *PQ*. Show your work.

MATH TIP

The *Pythagorean theorem* states that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs of the right triangle. In other words, for a right triangle with hypotenuse c and legs a and b, $c^2 = a^2 + b^2$.

6. Attend to precision. What is the distance between points $P(1, 4)$ and $Q(13, 9)$? How do you know?

7. How do you know that the triangle you drew in Item 2 is a right triangle?

8. What relationship do you notice among the coordinates of points *P*, *Q*, and *R*?

Check Your Understanding

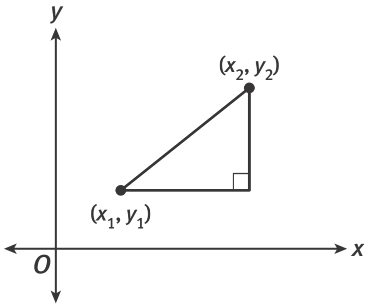
9. Explain how you would find the distance between points $J(-3, 6)$ and $K(3, 14)$.

10. What is the distance between $M(9, -5)$ and $N(-11, 10)$?

My Notes

Although the method you just learned for finding the distance between two points will always work, it may not be practical to plot points on a coordinate plane and draw a right triangle each time you want to find the distance between them.

Instead, you can use algebraic methods to ***derive*** a formula for finding the distance between any two points on the coordinate plane. Start with any two points (x_1, y_1) and (x_2, y_2) on the coordinate plane. Visualize using these points to draw a right triangle with a horizontal leg and a vertical leg.



11. What are the coordinates of the point at the vertex of the right angle of the triangle?

12. Write an expression for the length of the horizontal leg of the right triangle.

13. Write an expression for the length of the vertical leg of the right triangle.

14. Use the Pythagorean theorem to write an expression for the length of the hypotenuse of the right triangle.

ACADEMIC VOCABULARY

The word ***derive*** can mean to base one idea on another idea. When you derive a formula, you write a formula based on another another mathematical relationship by following a series of logical steps.

MATH TIP

In Item 12, you can think of the *x*-coordinates of the endpoints of the horizontal leg as values on a horizontal number line.

In Item 13, you can think of the *y*-coordinates of the endpoints of the vertical leg as values on a vertical number line.

My Notes

MATH TIP

Squaring a difference and then taking the square root has the same effect as taking the absolute value of the difference. Both result in nonnegative quantities. That means that the absolute value symbols are not needed in your formula to find the distance between two points.

CONNECT TO AP

The coordinate geometry formulas introduced in this activity are used frequently in AP calculus in a wide variety of applications.

15. Express regularity in repeated reasoning. Write a formula that can be used to find d, the distance between two points (x_1, y_1) and (x_2, y_2) on the coordinate plane.

16. Use the formula you wrote in Item 15 to find the distance between the points with coordinates $(12, -5)$ and $(-3, 7)$.

MATH TIP

A Venn diagram illustrates logical relationships. Think about the commonalities between the Pythagorean theorem and the Distance Formula. These similarities are placed in the overlapping circles of the Venn diagram. The differences are shown in the nonoverlapping parts.

Check Your Understanding

17. Find the distance between the points with the coordinates shown.
a. $(-8, 5)$ and $(7, -3)$
b. $(3, 8)$ and $(8, 3)$

18. Create a Venn diagram that compares and contrasts the Pythagorean theorem and the Distance Formula.

19. Reason abstractly. Suppose two points lie on the same vertical line. Can you use the Distance Formula to find the distance between them? Explain.

20. Write and simplify a formula for the distance d between the origin and a point (x, y) on the coordinate plane.

My Notes

LESSON 14-1 PRACTICE

For Items 21 and 22, find the distance between the points with the given coordinates.

21. (−8, −6) and (4, 10)

22. (5, 14) and (−3, −9)

23. **Reason quantitatively.** Explain how you know that your answer to Item 22 is reasonable. Remember to use complete sentences and words such as *and, or, since, for example, therefore, because of, by the,* to make connections between your thoughts.

24. Use your Venn diagram from Item 18 to write a RAFT.

Role: Teacher

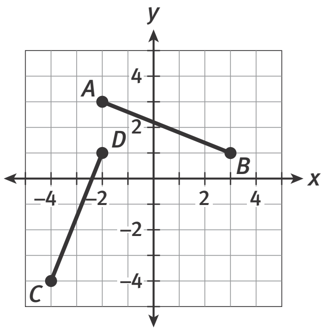
Audience: A classmate who was absent for the lesson on distance between two points

Format: Personal note

Topic: Explain the mathematical similarities and differences between the Pythagorean theorem and the Distance Formula.

25. The vertices of $\triangle XYZ$ are $X(-3, -6)$, $Y(21, -6)$, and $Z(21, 4)$. What is the perimeter of the triangle?

26. Use the Distance Formula to show that $\overline{AB} \cong \overline{CD}$.



ACADEMIC VOCABULARY

The ***format*** of a piece of writing can refer to its organization or style. For example, personal notes, essays, reports, and printed advertisements are all different formats for writing.

My Notes

MATH TIP

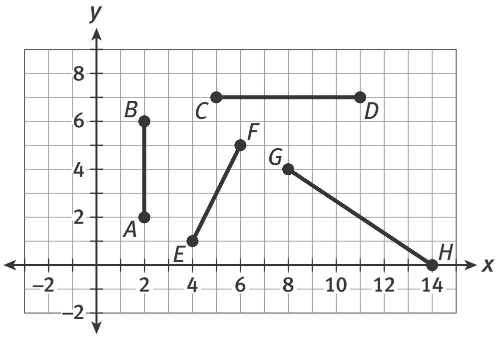
For a segment on a number line, the coordinate of the midpoint is the average of the coordinates of the endpoints.

Learning Targets:

- Use inductive reasoning to determine the Midpoint Formula.
- Use the Midpoint Formula to find the coordinates of the midpoint of a segment on the coordinate plane.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Visualization, Look for a Pattern, Debriefing, Work Backward, Self Revision/Peer Revision, Discussion Groups, Identify a Subtask

A *midpoint* is a point on a segment that divides it into two congruent segments. Follow the steps below to explore the concept of midpoint on the coordinate plane.



1. In the table below, write the coordinates of the endpoints of each segment shown on the coordinate plane.

Use appropriate tools strategically. Use a ruler to help you identify the midpoint of each segment. Then write the coordinates of the midpoint in the table.

Segment	Endpoints	Midpoint
$\overline{AB}$	A(____, ____) and B(____, ____)	(____, ____)
$\overline{CD}$	C(____, ____) and D(____, ____)	(____, ____)
$\overline{EF}$	E(____, ____) and F(____, ____)	(____, ____)
$\overline{GH}$	G(____, ____) and H(____, ____)	(____, ____)

2. Use the coordinates in the table in Item 1.

a. Compare the x-coordinates of the endpoints of each segment with the x-coordinate of the midpoint of the segment. Describe the pattern you see.

My Notes

b. Make use of structure. Compare the y-coordinates of the endpoints of each segment with the y-coordinate of the midpoint of the segment. Describe the pattern you see.

3. Use the patterns you described in Item 2 to write a formula for the coordinates of the midpoint M of a line segment with endpoints at (x_1, y_1) and (x_2, y_2).

MATH TIP

Your formula for the midpoint should have the form $M = (a, b)$, where a is an expression for the x-coordinate of the midpoint, and b is an expression for the y-coordinate of the midpoint.

Look back at the chart in Item 1 to verify the formula you wrote.

4. Use your formula to find the coordinates of the midpoint M of $\overline{AC}$ with endpoints $A(1, 4)$ and $C(11, 10)$. What can you conclude about segments AM and MC?

POINT OF INTEGRATION

Geometry and Algebra

When figures are plotted in the coordinate plane, you can use the midpoint formula to find midpoints of segments and the distance formula to find lengths of segments. Both formulas can be used to prove statements about geometric figures.

Check Your Understanding

5. Explain how you could check that your answer to Item 4 is reasonable.

6. Suppose a segment on the coordinate plane is vertical. Can you use the Midpoint Formula to find the coordinates of its midpoint? Explain.

7. The midpoint M of $\overline{ST}$ has coordinates (3, 6). Point S has coordinates (1, 2). What are the coordinates of point T? Explain how you determined your answer.

8. Reason quantitatively. The origin, (0, 0), is the midpoint of a segment. What conclusions can you draw about the coordinates of the endpoints of the segment?

My Notes

LESSON 14-2 PRACTICE

Find the coordinates of the midpoint of each segment with the given endpoints.

9. $Q(-3, 14)$ and $R(7, 5)$

10. $S(13, 7)$ and $T(-2, -7)$

11. $E(4, 11)$ and $F(-11, -5)$

12. $A(-5, 4)$ and $B(-5, 18)$

13. Find and explain the errors that were made in the following calculation of the coordinates of a midpoint. Then fix the errors and determine the correct answer.

Find the coordinates of the midpoint M of the segment with endpoints R(−2, 3) and S(13, −7).

$$M = \left(\frac{-2+3}{2}, \frac{13+(-7)}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{6}{2}\right) = \left(\frac{1}{2}, 3\right)$$

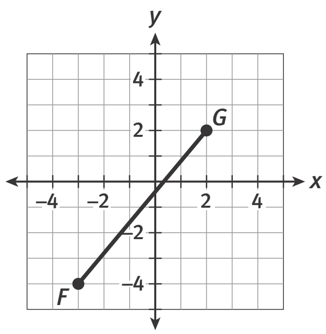
14. Make sense of problems. $\overline{HJ}$ is graphed on a coordinate plane. Explain how you would determine the coordinates of the point on the segment that is $\frac{1}{4}$ of the distance from H to J.

ACTIVITY 14 PRACTICE

Answer each item. Show your work.

Lesson 14-1

1. Calculate the distance between the points $A(-4, 2)$ and $B(15, 6)$.
2. Calculate the distance between the points $R(1.5, 7)$ and $S(-2.3, -8)$.
3. Describe how to find the distance between two points on the coordinate plane.
4. To the nearest unit, what is FG?

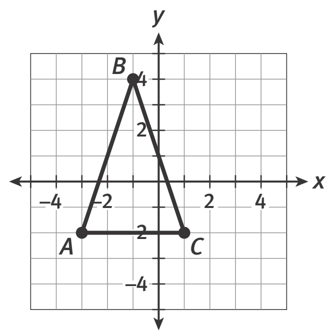


A. 5 units **B.** 6 units
C. 8 units **D.** 11 units

5. $\overline{JK}$ has endpoints $J(-3, -1)$ and $K(0, 3)$. $\overline{RS}$ has endpoints $R(1, 1)$ and $S(4, 4)$. Is $\overline{JK} \cong \overline{RS}$? Explain how you know.
6. The vertices of $\triangle LMN$ are $L(7, 4)$, $M(7, 16)$, and $N(42, 4)$.
   a. Find the length of each side of the triangle.
   b. What is the perimeter of the triangle?
   c. What is the area of the triangle? Explain how you determined your answer.
7. The distance from the origin to point P is 5 units. Give the coordinates of four possible locations for point P.
8. On a map, a trailhead is located at (5, 3) and a turnaround point is located at (15, 6). Each unit on the map represents 1 km. Ana and Larissa start at the turnaround point and walk directly toward the trailhead. If they walk at an average speed of 4.5 km/h, will they make it to the trailhead in less than 2 hours? Support your answer.

For Items 9–11, determine the length of each segment with the given endpoints.

9. $C(1, 4)$ and $D(11, 28)$
10. $Y(-2, 6)$ and $Z(5, -8)$
11. $P(-7, -7)$ and $Q(9, 5)$
12. Point R has coordinates (1, 3), and point S has coordinates $(6, y)$. If the distance from R to S is 13 units, what are the possible values of y?
13. Use the Distance Formula to show that $\triangle ABC$ is isosceles.



14. Draw a scalene triangle on a coordinate plane, and use the Distance Formula to demonstrate that your triangle is scalene.

Lesson 14-2

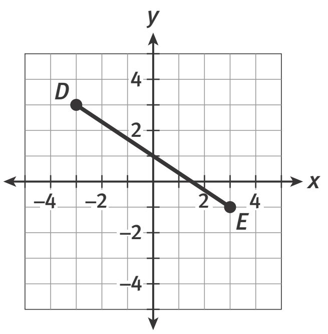
15. Determine the coordinates of the midpoint of the segment with endpoints $R(3, 16)$ and $S(7, -6)$.

16. Determine the coordinates of the midpoint of the segment with endpoints $W(-5, 10.2)$ and $X(12, 4.5)$.

17. Point C is the midpoint of $\overline{AB}$. Point A has coordinates (2, 4), and point C has coordinates (5, 0).
a. What are the coordinates of point B?
b. What is AB?
c. What is BC?

18. $\overline{JL}$ has endpoints $J(8, 10)$ and $L(20, 5)$. Point K has coordinates (13, 9).
a. Is point K the midpoint of $\overline{JL}$? Explain how you know.
b. How could you check that your answer to part a is reasonable?

19. Find the coordinates of the midpoint of $\overline{DE}$.



20. What are the coordinates of the midpoint of the segment with endpoints at (−3, −4) and (5, 8)?

A. (1, 2) **B.** (2, 4)
C. (4, 6) **D.** (8, 12)

21. A circle on the coordinate plane has a diameter with endpoints at (6, 8) and (15, 8).
a. What are the coordinates of the center of the circle?
b. What is the diameter of the circle?
c. What is the radius of the circle?
d. Identify the coordinates of another point on the circle. Explain how you found your answer.

22. Two explorers on an expedition to the Arctic Circle have radioed their coordinates to base camp. Explorer A is at coordinates (−26, −15). Explorer B is at coordinates (13, 21). The base camp is located at the origin.
a. Determine the linear distance between the two explorers.
b. Determine the midpoint between the two explorers.
c. Determine the distance between the midpoint of the explorers and the base camp.

23. A segment has endpoints with coordinates $(a, 2b)$ and $(-3a, 4b)$. Write the coordinates of the midpoint of the segment in terms of a and b.

24. Point J is the midpoint of $\overline{FG}$ with endpoints $F(1, 4)$ and $G(5, 12)$. Point K is the midpoint of $\overline{GH}$ with endpoints $G(5, 12)$ and $H(-1, 4)$. What is JK?

MATHEMATICAL PRACTICES
Look For and Express Regularity in Repeated Reasoning

25. Let (x_1, y) and (x_2, y) represent the coordinates of the endpoints of a horizontal segment.
a. Write and simplify a formula for the distance d between the endpoints of a horizontal segment.
b. Write and simplify a formula for the coordinates of the midpoint M of a horizontal segment.